The Pi-Phi Product

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The product of pi, $\pi=3.1415\ldots$, and phi, the Golden Ratio: $\phi= (1 + \sqrt{5})/2$, can be interpreted as the circumference of a circle with a diameter of phi. Is this the 'Golden Circle'? Huntley, in his book *The Divine Proportion* (Dover, 1970), encourages us to show that the curved surface area of a cone, with this circular base and a slant side length of one unit, is $\pi\phi/2$ square units. If two of these cones had their bases fused together then this "Golden Object" would have a curved surface area of $\pi\phi$ square units. He also informs us that the Golden Ellipse, minor axis of unit length and major axis of length phi, encloses a region of $\pi\phi$ square units. A circle with a radius equal to the geometric mean of the major and minor axes of the Golden Ellipse, i.e. the square root of the Golden Ratio, would also enclose an area of $\pi\phi$ square units. Should this circle be considered the Golden Circle? Our preference for the title Golden Circle would be the former, but it is the pi-phi product that we are interested in this note.

We have found that the pi-phi product can be expressed as four times the limit of an infinite series, S_B:

$$S_B = 1 + \sum_{k=1}^{\infty} a_k \{ 1/(F_{2k-1} + \phi F_{2k})\} \qquad \ldots 1$$

where F_k represent the Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_3 = 2,\ldots$), $a_k = b_k + c_k$, $b_k = (-1)^k/(2k+1)$, and $c_k = 0$ except when $k=3m+1$, $(m=0,1,\ldots)$ then $c_{3m+1} = b_m$. We have attempted to find this relationship in the literature but to date have been unsuccessful. We did notice a couple of infinite series involving the reciprocal of the Fibonacci numbers listed in a text by Vajda , *Fibonacci and Lucas Numbers and the Golden Section* (Halsted Press, 1989) but these turned out to have limits of phi but pi nowhere in sight:

$$4 - \sum_{k=0}^{\infty} (1/F_{2\wedge k}) = \phi = 1 + \sum_{k=2}^{\infty} (-1)^k \{ 1/ F_k F_{k-1}\} \qquad \ldots 2$$

Vajda also provides an infinite series involving the sum of arctangent functions each of which contains the reciprocal of a Fibonacci number with an odd numbered index. This sum converges to one quarter of pi:

$$\pi/4 = \sum_{k=1}^{\infty} \arctan(1/F_{2k+1}) \qquad \ldots 3$$

Clearly, infinite series containing reciprocals of the Fibonacci numbers exist but these do not converge to the pi-phi product. If it turns out that the sum S_B has not been observed, we would like to call it the Biwabik Sum in honor of the small northern Minnesota town in which both of us were raised. But where did this Biwabik Sum come from?

One of us (Oberg) noticed an interesting relationship between pi and phi while contemplating geometric questions related to the location of the King and Queen's burial chambers in the Great Pyramid (Cheops of Giza, Egypt). The essence of his geometric construction plan for the pyramid is shown in Fig. 1. The geometry leads to a pyramid angle of $\varphi = \arctan(\sqrt{\phi})$ or $\varphi = 51.827°$ (51°49'38"). This angle shows up in ancient diagrams involving the Golden Ratio (Lawlor, *Scared Geometry: Philosophy and Practice*, Thomas and Hudson, 1982) and seems to be a commonly accepted theoretical value for the pyramid angles (Banks, *Slicing pizzas, Racing Turtles and Further Adventures in Applied Mathematics*, Princeton University Press, 1999). The value is in close agreement with the measured value of $\varphi = 51.838°$ (51°50'15") determined from the dimensions of the Great Pyramid, base = 756 ft. and height = 481 ft., as given by Gillings, *Mathematics in the Time of the Pharaohs*, MIT Press, 1972. As an aside, Gillings also uncovers a theoretical value of a hypothetical pyramid angle in problem 56 of the Rhind Mathematical Papyrus, i.e. $\varphi = \arctan(7 \text{ palms}/ (5 + 1/25) \text{ palms}) = 54.246°$ (54°14'46"). A slight change in the geometry of Oberg's construction, Fig. 1f, leads to $\varphi = \arctan(\,(1 + \sqrt{(8\phi-11)(13-8\phi)}\,)/(8\phi-12)\,) = 54.609°$ (54°36'33").

Out of this quest to rationalize the design of the Great Pyramid came a diagram that ultimately led to the so called Oberg Formula, Fig 2. The other one of us (Johnson) noted that this formula could be obtained from an arctangent expression attributed to Euler (see Beckmann, *A History of Pi* (Golem, 1977), p.154), Fig. 3., by inserting $p = 1$ and $q = 1 + \phi$ into the Euler expression. Parenthetically, the same Euler expression can be used to derive eq. 3 by letting $p = F_{2k}$ and $q = F_{2k+1}$.

Since a sequence of arctangent relationships follow:

$$\arctan(1/1) = \arctan(1/2) + \arctan(1/3),$$
$$\arctan(1/3) = \arctan(1/5) + \arctan(1/8),$$
$$\arctan(1/8) = \arctan(1/13) + \arctan(1/21), \ \ldots$$

these can be combined to produce eq. 3.

Getting back to the Oberg Formula, it is simply a matter of using the Gregory series for the arctangent function, employing few identities involving the Golden Ratio and Fibonacci Numbers and the Biwabik Sum emerges quite naturally:

$$\pi\,\phi = 2^2\Big\{1 + \big[(2/3)/(F_1+F_2\phi) + (1/5)/(F_3+F_4\phi) - (1/7)/(F_5+F_6\phi)\big]$$
$$- \big[(2/9)/(F_7+F_8\phi) + (1/11)/(F_9+F_{10}\phi) - (1/13)/(F_{11}+F_{12}\phi)\big]$$
$$+ \big[(2/15)/(F_{13}+F_{14}\phi) + (1/17)/(F_{15}+F_{16}\phi) - (1/19)/(F_{17}+F_{18}\phi)\big]$$
$$- \ \ldots \Big\}$$

$$= 5.083203692....$$

…4

(the Biwabik Sum, S_B, is the expression between the curly braces)

It is clear that pi-phi product, the circumference of the proposed Golden Circle, is related to the Golden Ratio, the number two (the only even prime), the set of all odd numbers and the set of all Fibonacci numbers; each member of these two sets making a single appearance in the sum.

It would be aesthetically pleasing if the pi-phi product could be calculated in terms of only whole numbers. This can be done by replacing the Golden Ratio in the Biwabik Sum with $P_{n+1} = P_n - 1/F_{2^\wedge n}$ ($P_0 = 4$) and defining the partial sums by

$$S_n = 1 + \sum_{k=1}^{n} a_k \{ 1/(F_{2k-1} + P_n F_{2k}) \}$$

…5

(where the a_k's are the same as those in eq. 1). Now the limit of these partial sums will be $\pi\phi/4$ as n goes to infinity.

Finally, we note that the Oberg formula is a special case of a more general problem. The Euler arctan formula with $p = 1$ and $q = (1/z)^n - 1$ becomes, after defining $(1/z)^m = (z^n-1)/(z^n+1)$:

$$\pi/4 = \arctan((1/z)^n) + \arctan((1/z)^m) \qquad \ldots 6$$

As a consequence, for any combination of the whole numbers n and m, the roots, $z = z_0(n,m)$, of the polynomial $z^{m+n} - z^m - z^n - 1$ will produce two angles whose sum will be $\pi/4$, i.e. $\pi/4 = \theta_1 + \theta_2$ where $\tan\theta_1 = (1/z_0)^n$ and $\tan\theta_2 = (1/z_0)^m$. From this relationship it is easy to see that there are a countably infinite number of infinite series, the terms of which would have the form $A_k/(z_0(n,m))^k$ with the constants A_k being determined by the superposition of like powers of z_0 in the two Gregory series expansions for the arctangent functions. Roots for a few n and m combinations are shown graphically in Fig. 4. The actual numerical values are given in Table 1 which were found using Newton's method of locating roots. For three cases, we found simple expressions to compute the values for the roots: $z_0(1,1) = 1+ \sqrt{2}$, $z_0(1,2) = (1 + s_1 + s_2)/3$ with $s_{1,2} = (19$ +or- $3\sqrt{33})$^(1/3) and $z_0(1,3) = (1 + \sqrt{5})/2 = \phi$ (Oberg case) but have been unable to find expressions for the other roots.

In this note we have shown that the pi-phi product can be expressed as an infinite series involving the Fibonacci numbers, the Golden Ratio and the odd integers. A sequence of partial sums involving only the odd integers and Fibonacci numbers can also be shown to yield the pi-phi product in the limit. The series was obtained from the Oberg formula which turns out to be a particular case of a broader class of problems dealing with the roots of a polynomial of a certain form. We have no idea whether or not there may be other interesting aspects of this broader class of problems. It appears that phi is the root only for the n=1 and m=3 case, but does it show up as part of a factor of other roots, such as $a+b\phi$? Are there any rational number solutions for the roots? Or can the values of the roots be determined by relatively simple expressions like the three we found? We do not have the answers to these questions.

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Table 1 Some roots, $z_0(n,m)$, of the polynomial $z^{n+m} - z^m - z^n - 1$ and corresponding angles: $\cot(\theta_1)=(z_0)^n$ and $\cot(\theta_2)=(z_0)^m$

n	m	z_0	θ_1 deg	θ_1 min	θ_1 sec	θ_2 deg	θ_2 min	θ_2 sec
1	1	2.41421[1]	22	30	00	22	30	00
1	2	1.83929[2]	28	31	57	16	28	03
	3	1.61803[3]	31	43	03	13	16	57
	4	1.49709	33	44	29	11	15	31
	5	1.41963	35	09	40	09	50	20
2	3	1.42911	26	05	16	18	54	44
	4	1.35620	28	31	57	16	28	03
	5	1.30740	30	19	46	14	40	14
3	4	1.28845	25	03	24	19	56	36
	5	1.25207	26	59	49	18	00	11
4	5	1.21728	24	29	12	20	30	48

(1) (1 + √2)

(2) (1 + (19 + 3√33)^(1/3) + (19 - 3√33)^(1/3)) / 3

(3) (1 +√5) / 2

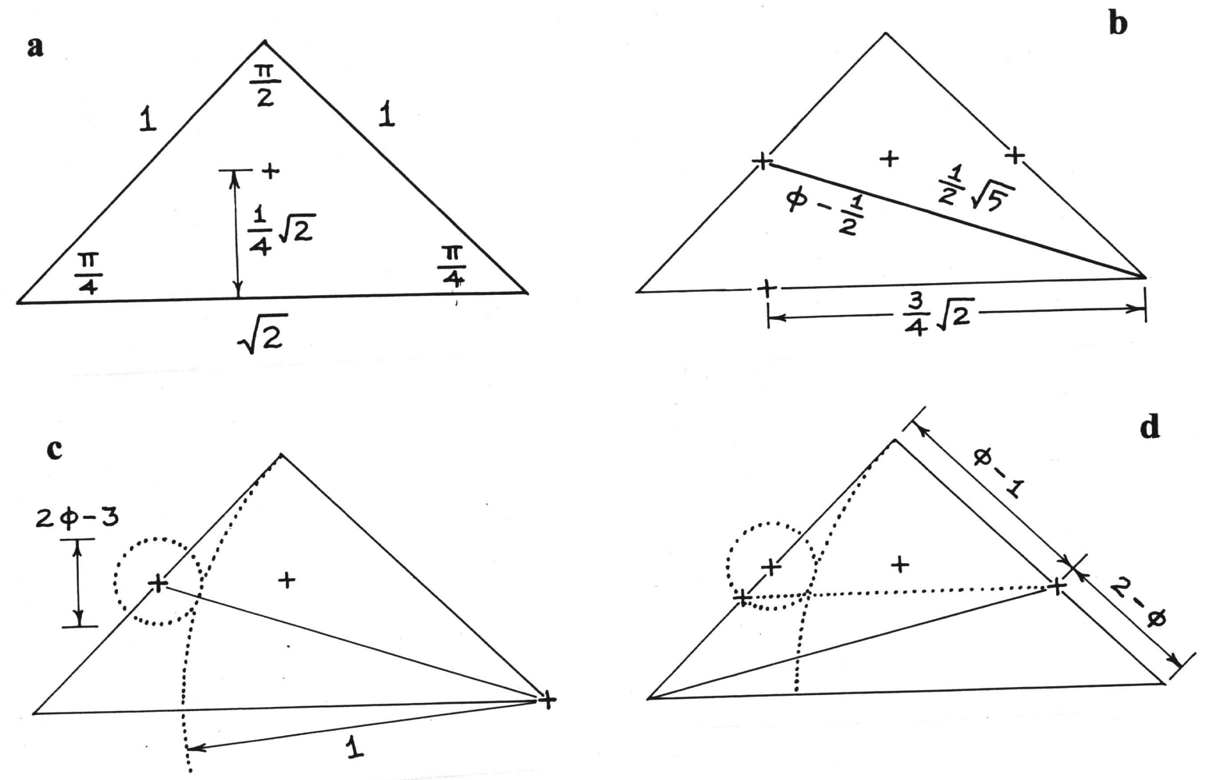


Fig. 1 The Great Pyramid: a Construction Plan

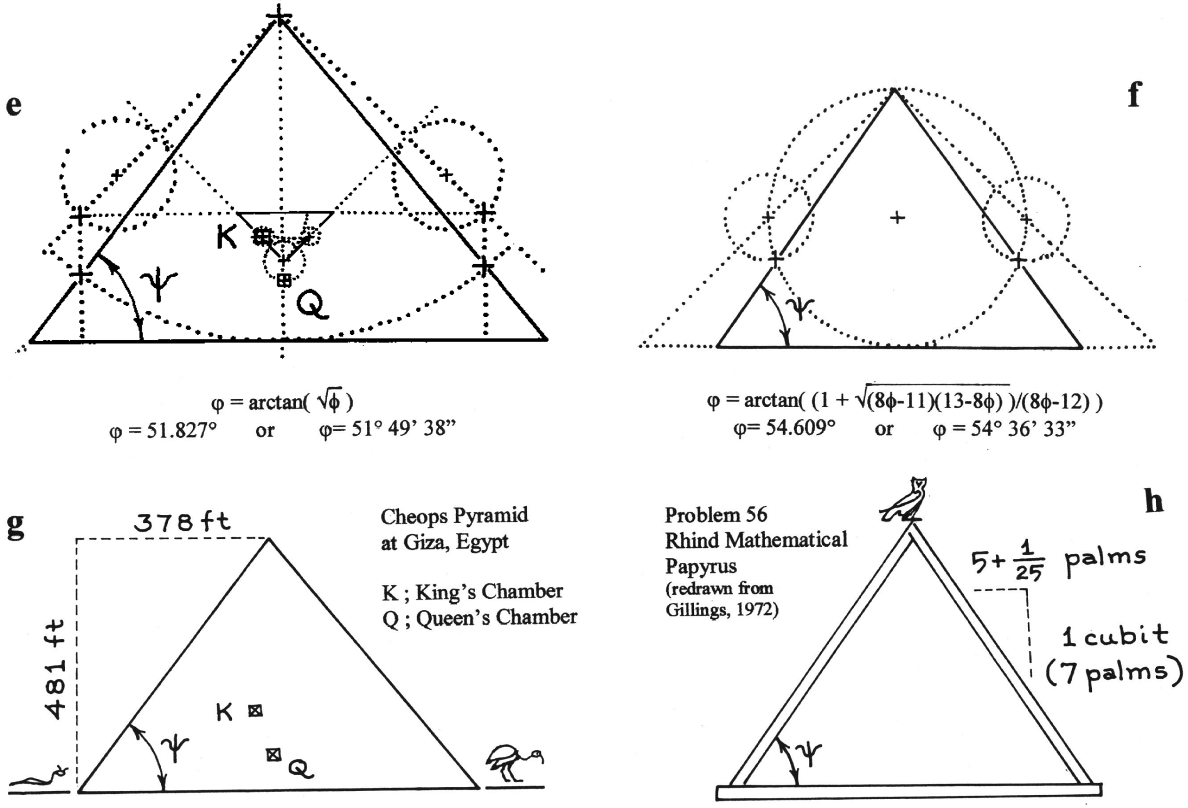


Fig. 1 The Great Pyramid: a Construction Plan (cont'd)

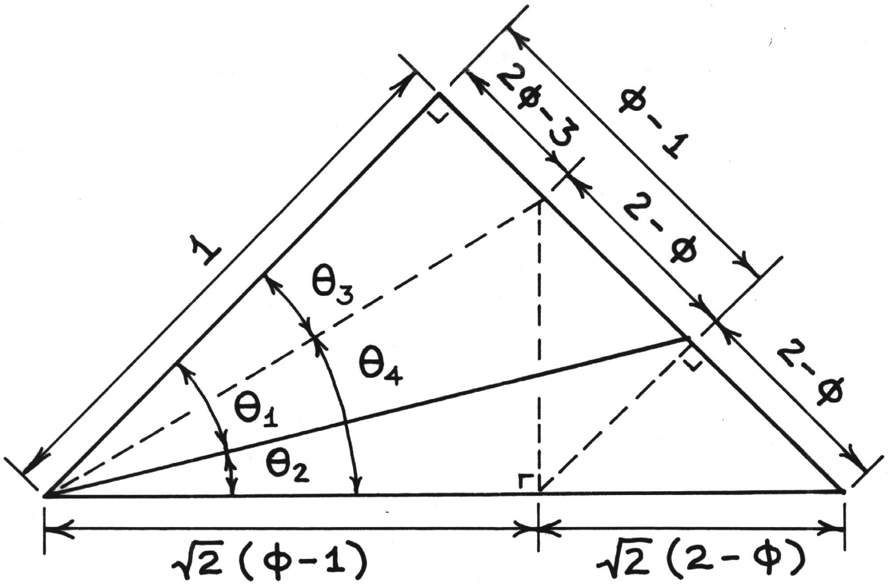
(e) $\varphi = \arctan(\sqrt{\phi})$; $\varphi = 51.827°$ or $\varphi = 51° 49' 38''$

(f) $\varphi = \arctan((1 + \sqrt{(8\phi-11)(13-8\phi)})/(8\phi-12))$; $\varphi = 54.609°$ or $\varphi = 54° 36' 33''$

(g) $\varphi = \arctan(481/378)$; $\varphi = 51.838°$ or $\varphi = 51° 50' 15''$

(h) $\varphi = \arctan(7.00/5.04)$; $\varphi = 54.246°$ or $\varphi = 54° 14' 46''$

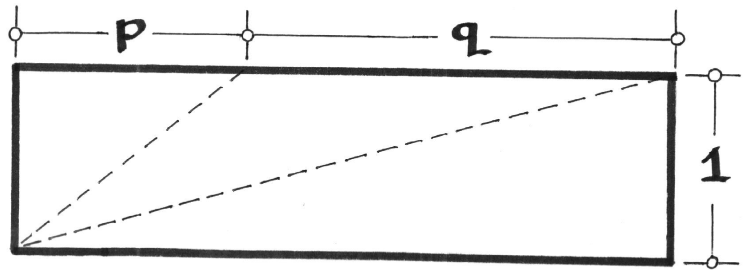
$$\theta_1 + \theta_2 \quad = \quad \theta_3 + \theta_4$$
$$\tan(\theta_1) = \phi - 1 = \quad 1/\phi \quad = (2-\phi)/(\phi-1) = \tan(\theta_4)$$
$$\tan(\theta_3) = 2\phi - 3 = \quad 1/\phi^3 \quad = \tan(\theta_2)$$



$$\pi/4 = \arctan(1) = \arctan(1/\phi) + \arctan(1/\phi^3)$$

Fig. 2 Oberg's Formula

(enlargement of Fig.1d)



$\arctan(1/p) = \arctan(1/(p+q)) + \arctan(q/(p^2+pq+1))$

Fig. 3 The Euler arctangent formula

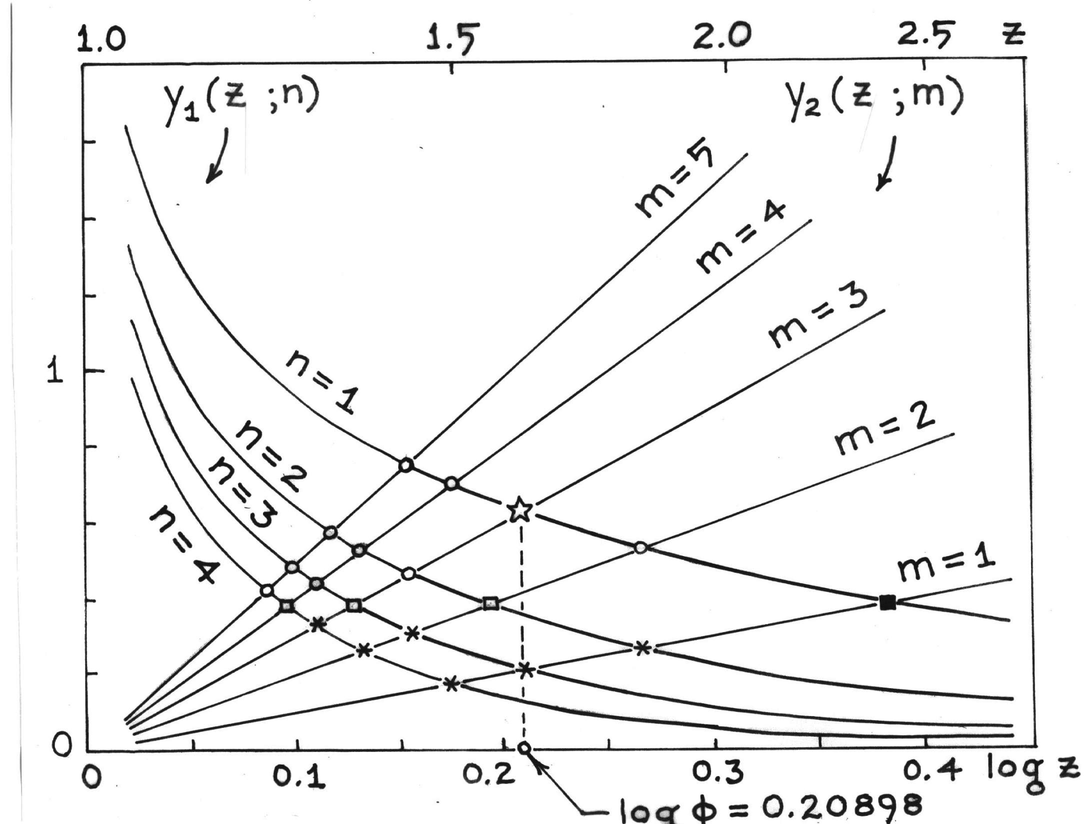


Fig. 4 log[$y_1(z; n)$] & log[$y_2(z; m)$] vs log z with $y_1 = (z^n + 1)/(z^n - 1)$ and $y_2 = z^m$

legend: o = $z_0(n,m)$, □ = $z_0(n,n)$, ✱ = $z_0(m,n)$ ☆ = $z_0(1,3) = \phi$ *(Oberg's formula)*

(note: $z_0(n,m) = z_0(m,n)$ and $z_0(n,n) = z_0(1,1)$^(1/n))